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A Markov Chain Based No-claim-discount System

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Abstract

In automobile insurance, among other general insurance policies, it is quite common to reduce the premium by a factor in case the insured does not make any claim in a given period. This is popularly known as NCD or no-claim-discount. Equally popular is the practice of increasing (known as 'loading') the premium, in case a claim is made. In effect, either system amounts to a multi-layer premium policy, where any particular policyholder is required to pick the level depending on the history of claims (s)he made in the immediate past few years. In this study, we inspect the desirability of this multi-layer premium system (also referred to as NCD system); to start with we work with a given number of levels with fixed gaps, and eventually look to determine an appropriate number of levels, as well the optimum gaps between levels.

The basic framework that we consider is that of a discrete time parameter Markov chain, where the state space consists of the different levels of the premium, and the state of a particular insured shifts randomly from an year to the next. The randomness of the transition is governed by the transition probability of causing an accident in a given year. We model this probability to be varying depending on quality of the driver. For the most part, we would be considering a finitely many groups of policyholders (drivers) characterized by respective probabilities of getting involved in an accident. The claim (damage) amount is assumed to have a known distribution, a log-normal one, for the sake of illustration. The parameters of this distribution need to be consistent with the basic premium level(s). We work under two possible behaviour of all the policyholders. In the simpler version, we consider a driver would claim whenever (s)he incurs a damage from accident. In the second framework, (s)he would make a claim only if the (claim) damage amount exceeds the potential loss in the form of losing NCD. In making this judgment (s)he would need to take into account the effect of any more accident in the near future, and it is assumed that (s)he would rule out the such eventuality. This assumption is reasonable in the sense that, in reality, most drivers think very positively about their own driving ability; also, one is likely to be extra careful after getting involved in an accident.

We obtain the stationary distribution for each group of policyholders. This reflects the distribution of a particular group over the various levels of premium in the long run. For example, one can obtain the percentage of 'good' drivers expected to receive the fully discounted rate in the long run. A comparative study of these stationary distributions over the various groups considered, form the basis of appropriateness of the assumed NCD system. We show the existence of a stationary distribution, irrespective of the possible differences in the various NCD systems.

Finally, we look at the feasibility of evolving a NCD system which would be fair in the sense that the premium collected from any specific group would cover the expected claim amount from that group asymptotically. Use of simulation, properties of Markov chain, along with the basic actuarial and statistical principles form the core of the methodologies adopted for this study.

A Markov Chain Based No-claim-discount System

Shubhabrata Das[1] and *Sankarshan Basu*[2]

1. Introduction

The premium on automobile insurance policies is reduced by a certain factor if no claim is made on that policy during the given period. This philosophy of reduced premium is also extended to some other general insurance policies, but is most widely prevalent amongst automobile insurance policies. This feature is popularly known as "No Claims Discount" or in short NCD. In simpler terms, it is nothing but the insurance providers way of rewarding the policyholders for being a careful driver (in the case of automobile insurance). Equally popular is the practice of increasing (known as 'loading') the premium, in case a claim is made. In affect, either system amounts to a multi-layer premium policy, where any particular policyholder is required to pick the level depending on the history of claims made by the policyholder in the immediate past few years. In this study, we inspect the desirability of this multi-layer premium system (also referred to as NCD system); to start with we work with a given number of levels with fixed gaps, and eventually look to determine an appropriate number of levels, as well the optimum gaps between levels. Though this system can be used (and is used) in various areas of general insurance, for the purpose of discussion here we shall restrict it to the case of automobile insurance. The results obtained for the automobile insurance case can be easily extrapolated to other areas of general insurance where this might be used.

The basic framework that we consider is that of a discrete time parameter Markov chain, where the state space consists of the different levels of the premium, and the state of a particular insured shifts randomly from an year to the next. The randomness of the transition is governed by the transition probability of causing an accident in a given year. We model this probability to be varying depending on quality of the driver. For the most part, we would be considering a finitely many groups of policyholders (drivers) characterized by respective probabilities of getting involved in an accident. The claim (damage) amount is assumed to have a known distribution, a log-normal one, for the sake of illustration. The parameters of this distribution need to be in consistence with the basic premium level(s). We work under two possible behaviours of all the policyholders. In the

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simpler version, we consider a policyholder would claim whenever damage is incurred due to an accident. In the second framework, the policyholder would make a claim only if the (claim) damage amount exceeds the potential loss in the form of losing NCD. In making this judgment the policyholder would need to take into account the effect of any more accidents in the near future, and it is assumed that the policyholder would rule out such eventualities. This assumption is reasonable in the sense that, in reality, most drivers think very positively about their own driving ability; also, one is likely to be extra careful after getting involved in an accident.

We divide the whole set of policyholders into two sets: good drivers and bad drivers and then obtain the stationary distribution for each group of policyholders separately. This reflects the distribution of a particular group over the various levels of premium in the long run. For example, one can obtain the percentage of 'good' drivers expected to receive the fully discounted rate in the long run. A comparative study of these stationary distributions over the various groups considered, form the basis of appropriateness of the assumed NCD system. We show the existence of a stationary distribution, irrespective of the possible differences in the various NCD systems.

Finally, we look at the feasibility of evolving a NCD system which would be fair in the sense that the premium collected from any specific group would cover the expected claim amount from that group asymptotically.

2. Literature Review

Many authors have studied NCD systems. A detailed description and comparison of most European and Japanese bonus-malus systems is found in Lemaire (1988). Recent studies in bonus-malus systems can be found in Lemaire (1995). Venter (1991) also looks at such systems, but primarily from an American perspective. Description of such systemes, their categories, such as discount levels and time required to deserve such discounts are highlighted in Borch (1990) and Verall (1995). Both Borch and Verall also give examples of transition matrices among categories as well as the limiting behaviour of such systems. Verall (1995) also discusses methods to approximate threshold values.. Discussion on bounds of optimal threshold values for bonus systems is also given in Goovaerts, et. al. (1984). In fact, Goovaerts also discusses methods of determining the bounds even when information on claims distribution is incomplete as well as obtaining the threshold values by backward induction. Another important work is that by De Pril (1979) where he considers a general bonus-malus system but with damages or rather claims following a

Poisson process. Other notable contributions in this area has been due to Norberg (1976), Loimaranta (1972), Pesonen (1963), Heras, Vilar and Gil (2001)

3. Stationary Distribution in a 3 stage NCD System

In this section, we discuss two possible scenarios that can happen in terms of movements by the drivers between the various levels of discount in the NCD system.

System 1: This is the first case when the movement is by one level. Here, the driver moves up by one level to a higher level of discount if no claim is made unless already in the highest level, in which case the driver stays at that level. Also, in case a claim is made then the driver moves down by one level to a lower level of discount unless already in the lowest level of discount (full premium), in which case the driver stays at that level. We refer to this as System 1.

Thus transition matrix (P) is given by:

Levels	1	2	3
1	p_{11}	$1-p_{11}$	0
2	p_{21}	0	$1-p_{21}$
3	0	p_{32}	$1-p_{32}$

The stationary solution in this case in given by solving the equation $\underline{\pi}P = \underline{\pi}$, where $\underline{\pi} = (\pi_1, \pi_2, \pi_3)$ and this results in the following values:

$$\pi_1 = \frac{p_{21}p_{32}}{(1-p_{11})(1+p_{32}) - p_{21}(1-p_{11}-p_{32})}, \quad \pi_2 = \frac{(1-p_{11})}{p_{21}}\pi_1, \quad \pi_3 = 1-\pi_1-\pi_2.$$

System 2: The second case is not as symmetric as the first case defined above. In this case the part of movement when no claim is made remains the same i.e. the driver moves up by one level to a higher level of discount if no claim is made unless already in the highest level, in which case the driver stays at that level. However, in case a claim is made the driver moves down to the highest level of claim (say level 1) irrespective of where the driver was at the time of making the claim. We refer to this as System 2.

Thus transition matrix (P) is given by:

Levels	1	2	3
1	p_{11}	$1-p_{11}$	0
2	p_{21}	0	$1-p_{21}$
3	p_{31}	0	$1-p_{32}$

The stationary solution in this case in given by solving the equation $\underline{\pi}P = \underline{\pi}$, where $\underline{\pi} = (\pi_1, \pi_2, \pi_3)$ and this results in the following values:

$$\pi_1 = \frac{p_{31}}{(1-p_{11})(1-p_{21}+p_{31})+p_{31}}, \quad \pi_2 = (1-p_{11})\pi_1, \quad \pi_3 = 1-\pi_1-\pi_2.$$

4. An Illustrative Example

In this section we describe through an example one possible NCD scheme – in particular the scheme defined in System 1. Based on the results and data obtained from this example, we then discuss the various features of the scheme and what variation can be made to make it better. Let us consider the following points:

- There are two kinds of drivers: say 10,000 'good' and 5,000 'bad'.
- For any driver, probability of making more than 1 accident in a year is negligible.
- There are three levels or categories of discount.
- On remaining accident – free in a year, a driver move up the ladder of categories by one (unless already in level 3).
- Similarly, on causing an accident, a driver move down the ladder of categories by one (unless already in level 1).
- Also, let the discounts associated with the 3 levels be as: In level 1 the discount is 0% i.e. the driver has to pay a full premium; in level 2 the discount is 20% of the full premium i.e. the driver is accident free for at least one year and hence gets a 20% reduction in the premium amount; and in level 3 the discount is 40% of the full premium i.e. the driver is accident free for at least two consecutive years and hence gets a 40% reduction in the premium amount.
- For good drivers, the probability of one accident in 1 year is 0.1 and for bad drivers the probability of one accident in 1 year is 0.25.

Let us now look at the case of the good drivers and bad drivers separately. For the good drivers, the transition probability matrix (P_1) is given by:

Levels	1	2	3
1	0.1	0.9	0
2	0.1	0	0.9
3	0	0.1	0.9

So what happens to the good drivers in the long run? For that, we need to obtain the stationary distribution. Let us assume that $\underline{\pi}^G = (\pi_1^G, \pi_2^G, \pi_3^G)$ is the stationary distribution. We need to solve the equation

$$\underline{\pi}^G P_1 = \underline{\pi}^G.$$

Solving this equation, we get $\pi_1^G = 0.0110$, $\pi_2^G = 0.0989$ and $\pi_3^G = 0.8901$. In other words, in the long run 110 good drivers would be at level 1 and pay the full premium, 989 of them would be at level 1 and get a 20% discount on the full premium, while the remaining 8901 of them will be in level 3 and receive the maximal discount of 40%.

Now, similarly for the bad drivers: In that case, the transition probability matrix (P_2) is given by:

Levels	1	2	3
1	0.25	0.75	0
2	0.25	0	0.75
3	0	0.25	0.75

So what happens to the bad drivers in the long run? For that, we need to again obtain the stationary distribution for the bad drivers. Let us assume that $\underline{\pi}^B = (\pi_1^B, \pi_2^B, \pi_3^B)$ is the stationary distribution. We need to solve the equation

$$\underline{\pi}^B P_2 = \underline{\pi}^B.$$

Solving this equation, we get $\pi_1^B = 0.0769$, $\pi_2^B = 0.2308$ and $\pi_3^B = 0.6923$. In other words, in the long run 384.5 good drivers would be at level 1 and pay the full premium, 1154 of them would be at level 1 and get a 20% discount on the full premium, while the remaining 3461.5 of them will be in level 3 and receive the maximal discount of 40%.

Based on the facts stated above, one would obviously ask the question: Is the model satisfactory? Let us look at some of the issues highlighted by the data given above before we comment on that.

- A bad driver causes accident 2.5 times as often as a good driver.
- Total expected no of accidents per year = 2250, of which, about 5/9th are caused by bad drivers.
- On the average, good drivers would pay about 62.42% of the full premium; while bad drivers pay 67.69% of the full premium. This is only marginally higher

though the probability of causing an accident by a bad driver is 2.5 times that of a good driver.

- Good drivers in aggregate pay 64.84% of the total premium collected. This can be regarded as being highly unfair, especially when the bad drivers cause 55.56% of the accidents.

Thus, it is obvious that this kind of a setup is highly biased against the good drivers – just contrary to what an ideal NCD scheme should be. We now list some possible variations that could lead to an improved scheme. The possible variations are:

- Variation in the probability of accident between the groups is possibly not big enough
- A driver causing an accident would not necessarily claim unless the belief is that the reimbursement would offset the possible loss of NCD.
- Severity of accidents (magnitude of claims) should be brought into consideration.
- Maybe there should be higher number of categories as well as magnitudes of discounts as well as the number of groups or categories of drivers.

We now try to address these one by one.

The first thing that one tries to study is the effect of the difference in probability of accident for good and bad drivers and one tries to see is varying these probabilities do make a difference or not. Given below is a table that shows a number of possible values of the probability of accidents, the proportion of drivers in each of the categories of discount as well as the average premium paid.

P [Accident]	Proportion of drivers (in long run)			Average Premium
	Full Premium	20% discount	40%discount	
0.05	0.0026	0.0499	0.9475	0.6110
0.1	**0.0110**	**0.0989**	**0.8901**	**0.6242**
0.15	0.0258	0.1461	0.8281	0.6395
0.2	0.0476	0.1905	0.7619	0.6571
0.25	**0.0769**	**0.2308**	**0.6923**	**0.6769**
0.3	0.1139	0.2658	0.6203	0.6987
0.35	0.1586	0.2945	0.5469	0.7223
0.4	0.2105	0.3158	0.4737	0.7474
0.45	0.2691	0.3289	0.4020	0.7734
0.5	0.3333	0.3333	0.3333	0.8000
0.55	0.4020	0.3289	0.2691	0.8266
0.6	0.4737	0.3158	0.2105	0.8526
0.65	0.5469	0.2945	0.1586	0.8777
0.7	0.6203	0.2658	0.1139	0.9013
0.75	0.6923	0.2308	0.0769	0.9231

Thus, it is clear that the average premium paid increases with the increase in probability of accident, however not at any alarming rate, especially in the realistic range. Also, the stationary distribution across the three levels, namely

$$\pi_1 = \frac{p^2}{1+p^2-p}, \quad \pi_2 = \frac{1-p}{p}\frac{p^2}{1+p^2-p}, \quad \pi_3 = 1-\pi_1-\pi_2.$$

does not depend on the discount levels. This can be accounted for under the next consideration, where the size of the claim also appears in the picture.

Let us now look at the case where the claim is made only if benefit outweighs loss of NCD. For illustrative purposes, we make the following assumptions in addition to the ones made for the example above:

- Full premium is INR 10,000
- Claim size has a log-normal distribution with mean (μ) = 7 and standard deviation (σ) = 2.5.
- Time value of money is ignored
- All drivers (following an accident) believe that they will not have another accident in the next two years.

Based on these assumptions, the decision to make a claim or not shall be dependent on the following two factors:

- Size of the Claim
- Level to which the driver currently belongs

Thus, we can now look at the possible losses suffered by making a claim depending on the level at which the driver current is. One has to look at the potential losses that the river will suffer from not getting the NCD over the two-year period subsequent to making a claim (that is after having an accident). The potential loss for making a claim or the loss of NCD will be as follows:

Level	Action	Premium Next year	Premium the year after	NCD Loss
1	*Claim*	10,000	8,000	4,000
	Do not claim	8,000	6,000	
2	*Claim*	10,000	8,000	6,000
	Do not claim	6,000	6,000	
3	*Claim*	8,000	6,000	2,000
	Do not claim	6,000	6,000	

So, now the important question remains: ***When to claim?*** It is obvious that a claim shall be made only if the amount of claim (which without loss of generality can be assumed as the damage) is greater than the loss of NCD; other wise it might be better not to claim and make use of the NCD at the end of the year. Thus, writing this in probability terms, and using the fact that claims have a log-normal distribution with $\mu = 7$ and $\sigma = 2.5$, we have:

$$P[Claim \mid Accident] = P[Damage > NCD\ Loss] = P[\ln(Damage) > \ln(NCD\ Loss)]$$

$$= P\left[Z > \frac{\ln(NCD\ Loss) - \mu}{\sigma}\right]$$

Thus, if we look at the table given above on the amount of NCD Loss for a given level, we can easily find the P[Claim | Accident]. The figures are shown in the table below:

Level	NCD Loss	Probability [Claim | Accident]
1	4,000	**0.3024**
2	6,000	**0.2483**
3	2,000	**0.4050**

We would now want to determine the transition matrix for the good / bad driver depending on the current level. In other words, we would want to find out the probability of claim for a good / bad driver depending on the current level as this determines the transition matrix. This is given as:

$$P[Claim \mid Good\ Driver] = P[Accident \mid Good\ Driver] \times P[Claim \mid Accident]$$

where the second term on the right hand side would stay the same for the bad drivers as well (but may depend on the claim size and the level at which the driver currently is).

Using this result, we can then set up the transition probability matrix and obtain the stationary distributions for the good and bad drivers under the assumption that the claim will be made only if the damage is greater than the loss of NCD suffered by making the claim. The transition probability matrix (P_3) for the good driver is as follows:

Levels	**1**	**2**	**3**
1	0.0302	0.9698	0
2	0.0248	0	0.9752
3	0	0.0405	0.9595

Also, the stationary distribution $\underline{\pi}^G = (\pi_1^G, \pi_2^G, \pi_3^G)$ is obtained by solving the equation

$$\underline{\pi}^{G} P_3 = \underline{\pi}^{G}.$$

This results in the fact that $\pi_1^G = 0.0010$, $\pi_2^G = 0.0398$ and $\pi_3^G = 0.9591$. In other words, here there will be 10 good drivers at level 1 paying the full premium, 398 of them would be at level 1 and get a 20% discount on the full premium, while the remaining will be in level 3 and receive the maximal discount of 40% in the long run.

Now, similarly for the bad drivers: In that case, the transition probability matrix (P_4) is given by:

Levels	1	2	3
1	0.0756	0.9244	0
2	0.0621	0	0.9379
3	0	0.1013	0.8987

So what happens to the bad drivers in the long run? For that, we need to again obtain the stationary distribution for the bad drivers. Let us assume that $\underline{\pi}^{B} = (\pi_1^B, \pi_2^B, \pi_3^B)$ is the stationary distribution. We need to solve the equation

$$\underline{\pi}^{B} P_4 = \underline{\pi}^{B}.$$

Solving this equation, we get $\pi_1^B = 0.0065$, $\pi_2^B = 0.0968$ and $\pi_3^B = 0.8967$. In other words, in the long run 32.5 good drivers would be at level 1 and pay the full premium, 484 of them would be at level 1 and get a 20% discount on the full premium, while the remaining will be in level 3 and receive the maximal discount of 40%.

Let us now look at the summary of this system – the salient features that come out of this are as follows:

- It does not seem to work any better than the previous case where the size of the claim or level at which the driver is, was not taken into account. It does show that this case is slightly better for drivers in level 2 – but then this is true for both good and bad drivers.
- Expected premium for good drivers is 60.81% of the full premium for good drivers and 62.17% of the full premium for bad drivers. Thus, the premium for bad drivers is still being subsidized from the premium collected from the good drivers – something that good drivers might not be too comfortable with.

A couple of possible avenues out of this might be higher incentives to avoid accidents as well as a strategy to fairly share the cost between the good and bad drivers.

5. Desirable Properties of a NCD

We now try to look at some desirable properties of a NCD system and see how these affects the stationary distributions thus obtained as well as the discount level. Note that whatever the NCD system, the stationary distribution for the given transition probability matrix always exists; i.e. the equation $\underline{\pi}P = \underline{\pi}$, where $\underline{\pi}$ denotes the stationary distribution of the NCD system and P is the associated transition probability matrix.

Let us now assume that F denotes the full premium and D denotes the discount level. For easier understanding, we are assuming only two stages – that is either the driver pays a full premium F or a discounted premium $F(1 - D)$ in a given year. We can then try and obtain the stationary distribution for the two groups of drivers (good and bad) between the two levels. Also assumed is the fact that the claim size that would justify a claim by losing the NCD would have to be at least equal to or greater than FD and that the claims have a log-normal distribution with mean μ and variance σ^2.

As stated earlier, the stationary distribution always exists, and for either group, the proportion of drivers paying full premium (level 1) would be the same as the proportion among them who claims during a year. This is highly desirable property and should be kept in mind while designing a NCD system. The stationary solutions are then given as:

$$\pi_G = p_G \Phi\left(\frac{\mu - \ln(FD)}{\sigma}\right), \quad \pi_B = p_B \Phi\left(\frac{\mu - \ln(FD)}{\sigma}\right)$$

Note that here we have taken the mean and the variance of the claims for both good and bad drivers to be the same – however, that is not a necessary precondition and these values could be different.

Now, we impose an additional requirement – a reasonable assumption. We assume that the NCD system is run as a "*no – profit no – loss setup*". Thus, ignoring other costs, the total premium collected should be equal to the total expected payout from claims.

Under the stationary distribution (i.e. in the long run) with two groups of drivers, this leads to the following equation:

$$(N_G\pi_G + N_B\pi_B)F + \{N_G(1-\pi_G) + N_B(1-\pi_B)\}FD = (N_G\pi_G + N_B\pi_B)\exp(\mu + 0.5\sigma^2) \quad (*)$$

One of the goals of a NCD system could be to attain the balance this within each homogenous group. Thus, one would require:

$$(PC_G =)\ N_G \pi_G F + N_G (1 - \pi_G) FD = N_G \pi_G \exp(\mu + 0.5\sigma^2) \quad (= EP_G)$$
$$\text{and } PC_B = EP_B \qquad (**)$$

Here PC stands for 'premium collected' and EP for 'expected payout'. For illustration, we have assumed lognormal losses, but this assumption is not key to the approach adopted.

Based on this setting we come up with the result: ***The group wise balance equation (**) does not have any solution consistent with the stationary solution.***

Hence we work under overall no-profit solution and seek to minimize disparity between PC and EP across groups. We can then try and obtain either:

- The full premium F given a fixed value of the discount level D (for illustrative purposes we may assume the discount level to be 20%)
- The discount level D, given the full premium F – we do this at the average claim cost per individual basis; namely

$$F = (\frac{N_G \pi_G + N_B \pi_B)}{N_G + N_B} \exp(\mu + 0.5\sigma^2)$$

- Find F and D by minimizing

$$| PC_G - EP_G | + | PC_B - EP_B | \text{ subject to equation (*) and } 0 \le D \le 50\%.$$

- Find F and D by minimizing

$$\max\left(\left|\frac{PC_G}{EP_G} - 1\right|, \left|\frac{PC_B}{EP_B} - 1\right|\right) \text{ subject to equation (*) and } 0 \le D \le 50\%.$$

In the last two cases, 50% upper limit for discount is just for illustration, although there is clear case for setting a limit like this. Typically this may be a function of the number of discount levels.

Conjecture: The last two minimization problems shown above, may lead to an identical solution as is evident from a number of numerical trials.

6. Remarks

The proposed work needs to be extended covering multiple groups of drivers and levels than considered here. Also large scale simulation and computational exercise needs to be undertaken to obtain full validity of the work. It is also important to pay attention to the optimization algorithm so that the solution thus obtain is indeed a global optimal not a local one.

7. References

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